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A note on the classifications of hyperbolic and elliptic equations with polynomial coefficients

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Abstract

In this work we consider the hyperbolic and elliptic partial differential equations with constant coefficients; then by using double convolutions we produce new equations with polynomial coefficients and classify the new equations. It is shown that the classifications of hyperbolic and elliptic equations with non-constant coefficients are similar to those of the original equations; that is, the equations are invariant under double convolutions.

Keywords: Hyperbolic equation; Elliptic equation; Double convolution; Classification of PDE

1. Introduction

It is well known that the wave and Laplace's equations are fundamental equations in mathematical physics and occur in many branches of physics as well as in applied mathematics and engineering; see [1,2,4]. In this study we classify hyperbolic and elliptic equations after convolution, where the coefficients are considered as polynomials. First of all we give the following definition since it has a relation to our study.

Definition 1. If $F_1(x, y)$ and $F_2(x, y)$ are integrable functions, then the following integral:

$$F_1(\theta_1, \theta_2) ** F_2(\theta_1, \theta_2) = \int_0^y \int_0^x F_1(x - \theta_1, y - \theta_2) F_2(\theta_1, \theta_2) \mathrm{d}\theta_1 \mathrm{d}\theta_2$$

is called a double convolution, provided that the integral exists; see the details in [4].

2. Classification of second-order partial differential equations

Consider a second-order partial differential equation with non-constant coefficients in the form

$$a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} + d(x, y)u_x + e(x, y)u_y + f(x, y)u = 0 \tag{2.1}$$

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and an almost linear equation in two variables

$$au_{xx} + 2bu_{xy} + cu_{yy} + F(x, y, u, u_x, u_y) = 0 \tag{2.2}$$

where a, b, c, d, e, f are of class $C^2(\Omega)$, $\Omega \subseteq \mathbb{R}^2$ is the domain, $(a, b, c) \neq (0, 0, 0)$, and the expression $au_{xx} + 2bu_{xy} + cu_{yy}$ is called the principal part of Eq. (2.1); since the principal part mainly determines the properties of the solution, we shall classify the more general form of equation (2.2) instead of Eq. (2.1) by considering the function

$$d(x, y) = b^2(x, y) - a(x, y)c(x, y). \tag{2.3}$$

It is well known that we have

$d > 0$ Hyperbolic

$d < 0$ Elliptic

$d = 0$ Parabolic.

See [3].

Hyperbolic equations

First of all we consider the particular hyperbolic wave equation with non-constant coefficients

$$(p(x, t) ** a(x, t))u_{tt} - (p(x, t) ** c(x, t))u_{xx} = f_1(x, t) ** f_2(x, t) \tag{2.4}$$

where the symbol $**$ indicates double convolution and $p(x, t), a(x, t), c(x, t)$ are polynomials defined by

$$p(x, t) = \sum_{j=1}^{n} \sum_{i=1}^{m} x^i t^j, \qquad a(x, t) = \sum_{\beta=1}^{n} \sum_{\alpha=1}^{m} x^\alpha t^\beta, \qquad c(x, t) = \sum_{l=1}^{n} \sum_{k=1}^{m} x^k t^l \tag{2.5}$$

for $i \neq j, \alpha \neq \beta$ and $k \neq l$, $i, \alpha, k = 1, 2, 3 \ldots m$ and $j, \beta, l = 1, 2, 3 \ldots n$; it follows that the coefficients of equation (2.4) are given by

$$A(x, y) = p(x, t) ** a(x, t) = \sum_{j=1}^{n} \sum_{i=1}^{m} x^i t^j ** \sum_{\beta=1}^{n} \sum_{\alpha=1}^{m} x^\alpha t^\beta$$

and by using the double-convolution definition and integration by parts, we obtain

$$\begin{aligned} A(x, y) &= \sum_{j=1}^{n} \sum_{i=1}^{m} x^i t^j ** \sum_{\beta=1}^{n} \sum_{\alpha=1}^{m} x^\alpha t^\beta \\ &= \int_0^y \sum_{i=1}^{n} \sum_{\alpha=1}^{n} (t - \eta)^j \eta^\beta \mathrm{d}\eta \times \int_0^x \sum_{i=1}^{m} \sum_{\alpha=1}^{m} (x - \zeta)^i \zeta^\alpha \mathrm{d}\zeta. \end{aligned} \tag{2.6}$$

The first integral in the right hand side of Eq. (2.6) can be obtained as

$$\int_0^y \sum_{j=1}^{n} \sum_{\beta=1}^{n} (t - \eta)^j \eta^\beta \mathrm{d}\eta = \sum_{j=1}^{n} \sum_{\beta=1}^{n} \frac{j! t^{\beta+j+1}}{(\beta + 1)((\beta + 2) \cdots (\beta + j + 1))} \tag{2.7}$$

and in a similar way the second integral in the right hand side of Eq. (2.6) is given by

$$\int_0^x \sum_{i=1}^{m} \sum_{\alpha=1}^{m} (x - \zeta)^i \zeta^\alpha \mathrm{d}\zeta = \sum_{i=1}^{m} \sum_{\alpha=1}^{m} \frac{i! x^{\alpha+i+1}}{(\alpha + 1)((\alpha + 2) \ldots (\alpha + i + 1))}. \tag{2.8}$$

Thus from Eqs. (2.7) and (2.8) we obtain the first coefficient of Eq. (2.4) in the form

$$A(x, t) = \sum_{j=1}^{n} \sum_{\beta=1}^{n} \sum_{i=1}^{m} \sum_{\alpha=1}^{m} \frac{i! j! x^{\alpha+i+1} t^{\beta+j+1}}{(\alpha + 1)((\alpha + 2) \ldots (\alpha + j + 1))(\beta + 1)((\beta + 2) \ldots (\beta + j + 1))}. \tag{2.9}$$

Similarly, for the coefficient of the second part of Eq. (2.4) we have

$$C(x,t)=\sum_{j=1}^{n}\sum_{l=1}^{n}\sum_{k=1}^{m}\sum_{i=1}^{m}\frac{i!j!x^{k+i+1}t^{l+j+1}}{(k+1)((k+2)\ldots(k+i+1))(l+1)((l+2)\ldots(l+j+1))} \tag{2.10}$$

and then one can easily set up

$$D(x,t)=-A(x,t)C(x,t). \tag{2.11}$$

Now we can consider some particular cases.

(I) If $i=\alpha$, $j=\beta$ and $i=k, l=j$ are either even or odd numbers, Eq. (2.9) becomes

$$p(x,t)**a(x,t)=\sum_{j=1}^{n}\sum_{i=1}^{m}\frac{i!j!x^{2i+1}t^{2j+1}}{(i+1)((i+2)\ldots(2i+1))(j+1)((j+2)\ldots(2j+1))}, \tag{2.12}$$

Eq. (2.10) becomes

$$p(x,t)**c(x,t)=\sum_{j=1}^{n}\sum_{i=1}^{m}\frac{i!j!x^{2i+1}t^{2j+1}}{(i+1)((i+2)\ldots(2i+1))(j+1)((j+2)\ldots(2j+1))} \tag{2.13}$$

and Eq. (2.11) becomes

$$H(x,t)=\left[\sum_{j=1}^{n}\sum_{i=1}^{m}\frac{i!j!x^{2i+1}t^{2j+1}}{(i+1)((i+2)\ldots(2i+1))(j+1)((j+2)\ldots(2j+1))}\right]^2. \tag{2.14}$$

(II) If $i=\alpha=j=\beta=k=j$, then Eq. (2.9) gives

$$p(x,t)**a(x,t)=\sum_{i=1}^{n}\sum_{i=1}^{m}\frac{(i!)^2\,x^{2i+1}t^{2i+1}}{(i+1)((i+2)\ldots(2i+1))^2} \tag{2.15}$$

and similarly for Eq. (2.10). Then Eq. (2.11) becomes

$$C(x,t)=\left[\sum_{i=1}^{n}\sum_{i=1}^{m}\frac{(i!)^2\,x^{2i+1}t^{2i+1}}{(i+1)((i+2)...(2i+1))^2}\right]^2. \tag{2.16}$$

Now we have further cases.

Case (i): If $i+j, \alpha+\beta$ and $k+l$ are odd, we classify Eq. (2.4) by using Eq. (2.11); in the case of $b=0$ only, we check this equation.

$$D(x,t)=A(x,t)C(x,t) \tag{2.17}$$

where the powers of x and t in the polynomials $a(x,t)$ and $c(x,t)$ are either even or odd, and thus the powers of x, t in Eq. (2.11) are even; thus we see that $D>0$ for all $(x_{0,}t_0)\in\mathbb{R}^2$, and Eq. (2.4) is a hyperbolic equation.

Then if we consider a particular non-constant coefficient wave equation in one dimension in the form

$$(x^2t**x^6t^3)u_{tt}-(x^2t**x^4t^3)u_{xx}=e^{x+t} \tag{2.18}$$

and then consider the coefficients by using Eqs. (2.9) and (2.10), we get

$$A=x^2t**x^6t^3=\frac{1}{5040}x^9t^5$$

$$C=-x^2t**x^4t^3=-\frac{1}{2100}x^7t^5.$$

Then

$$D=-AC=\frac{1}{10\,584\,000}x^{16}t^{10}. \tag{2.19}$$

We can easily see from Eq. (2.19) that Eq. (2.18) is hyperbolic for all $(x_0, t_0) \in \mathbb{R}^2$.

Case (ii): If $i + j, \alpha + \beta$ and $k + l$ are even in Eq. (2.4), and we assume $b = 0$, then the powers of x, t in Eq. (2.11) are even; thus it follows that for all points (x_0, t_0) in the domain $\mathbb{R}^2$, Eq. (2.4) is hyperbolic.

Now if we consider another particular non-constant coefficient wave equation in one dimension in the form

$$(xt^3 ** x^5t^3)u_{tt} - (xt^3 ** x^7t^5)u_{xx} = \sin(x + t), \tag{2.20}$$

like in the above example we make calculations and obtain

$$D(x, t) = -AC = \frac{1}{213\,373\,440} x^{16}t^{16}. \tag{2.21}$$

From Eq. (2.21) we see that Eq. (2.20) is hyperbolic for all points (x_0, t_0) in the domain $\mathbb{R}^2$.

Case (iii): If $i + j$ is odd, $\alpha + \beta$ and $k + l$ are even in Eq. (2.4), then we are going to classify Eq. (2.4) by using Eq. (2.11); in this case $b = 0$ and therefore we only check Eq. (2.11). The powers of x, t in Eq. (2.11) are even. Then it follows that for all points (x_0, t_0) in the domain $\mathbb{R}^2$, Eq. (2.4) is hyperbolic. As a further particular case we consider a simple non-constant wave equation in one-dimensional form:

$$\left(x^3t^4 ** xt^5\right) u_{tt} - \left(x^3t^4 ** x^7t\right) u_{xx} = e^{x+t} ** \cos(x + t) \tag{2.22}$$

and as above we see that

$$D(x, t) = \frac{1}{997\,920\,000} 9x^{16}t^{16}. \tag{2.23}$$

Thus Eq. (2.23) is positive for all points $(x_0, t_0) \in \mathbb{R}^2$; thus Eq. (2.22) is hyperbolic.

Case (iv): If $i + j$ is even , $\alpha + \beta$ and $k + l$ are odd in Eq. (2.4), since the powers of x and t in polynomials $a(x, t)$, $c(x, t)$ are either even or odd, the powers of x, t in Eq. (2.11) are even; thus for all points (x_0, t_0) in the domain $\mathbb{R}^2$, Eq. (2.4) is hyperbolic.

In the next part we study the classification of non-constant coefficient elliptic equations.

Elliptic equations

We aim to examine the classification of elliptic equations; in particular, we consider equations with non-constant coefficients in the form

$$a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} = f_1(x, y) ** f_2(x, y) \tag{2.24}$$

where a, b and c are polynomials defined by

$$b(x, y) = \sum_{j=1}^{n} \sum_{i=1}^{m} x^s y^t, \qquad a(x, y) = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{m} p_{\alpha\beta} x^\alpha y^\beta, \qquad c(x, t) = \sum_{k=1}^{n} \sum_{l=1}^{m} q_{kl} x^k y^l. \tag{2.25}$$

Assume that $s + t, \alpha + \beta$ and $k + l$ are odd and Eq. (2.25) is to be an elliptic equation under the condition that the powers $s = \frac{\alpha+k}{2}$ and $t = \frac{\beta+l}{2}$ and the powers of x and y in the polynomials $a(x, y)$, $c(x, y)$ are either even or odd; also the coefficients of the two polynomials $a(x, y)$, $c(x, y)$ have the same sign. Now we are going to study the classification of Eq. (2.25). If we multiply by the polynomial $k(x, y)**$, this multiplication is a double convolution, where $k(x, y) = \sum_{j=1}^{n} \sum_{i=1}^{m} x^i y^j$, and Eq. (2.25) becomes

$$(k(x, y) ** a(x, y))\, u_{xx} + 2\,(k(x, y) ** b(x, y))\, u_{xy} + (k(x, y) ** c(x, y))\, u_{yy} = f_1(x, y) ** f_2(x, y) \tag{2.26}$$

where $i \neq j, s \neq t, \alpha \neq \beta$ and $k \neq l$. In the same way as we calculated the coefficients of hyperbolic equations, we calculate the coefficients of Eq. (2.26). The first coefficient is given by

$$A(x, y) = \sum_{j=1}^{n} \sum_{\beta=1}^{n} \sum_{i=1}^{m} \sum_{\alpha=1}^{m} \frac{\left(p_{\alpha\beta}\right) i!\,j!\,x^{\alpha+i+1} y^{\beta+j+1}}{(\alpha+1)((\alpha+2)\ldots(\alpha+i+1))(\beta+1)((\beta+2)\ldots(\beta+j+1))}, \tag{2.27}$$

the second coefficient is given by

$$B(x, y) = \sum_{j=1}^{n} \sum_{t=1}^{n} \sum_{i=1}^{m} \sum_{s=1}^{m} \frac{(q_{kl})\, i!j!x^{s+i+1} y^{t+j+1}}{(s+1)((s+2)\dots(s+i+1))(t+1)((t+2)\dots(t+j+1))} \tag{2.28}$$

and the last coefficient of Eq. (2.26) is given by

$$C(x, y) = \sum_{j=1}^{n} \sum_{l=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} \frac{(q_{kl})\, i!j!x^{k+i+1} y^{l+j+1}}{(k+1)((k+2)\dots(k+i+1))(l+1)((l+2)\dots(l+j+1))}. \tag{2.29}$$

From Eq. (2.30) below, we can check whether Eq. (2.26) is elliptic or not:

$$D(x, y) = B(x, y)^2 - A(x, y)C(x, y). \tag{2.30}$$

Now we have the following cases:

(i): Suppose that $i + j, s + t, \alpha + \beta$ and $k + l$ are odd and Eq. (2.26) is to be an elliptic equation under the condition that the powers $s = \frac{\alpha+k}{2}$ and $t = \frac{\beta+l}{2}$ and the powers of x and y in polynomials $a(x, y), c(x, y)$ are either even or odd; also the coefficients of the two polynomials $a(x, y), c(x, y)$ have the same sign. Now we are going to study the classification of Eq. (2.26). If we look at the powers of x, y in Eq. (2.30) we see that the power of $B(x, y)^2$ is equal to the power of $A(x, y)C(x, y)$ and the coefficient of $A(x, y)C(x, y) > 1$. Thus the power of (2.30) is even, and thus for all points (x_0, y_0) in the domain $\mathbb{R}^2$, Eq. (2.26) is an elliptic equation. In particular, if we consider as a simple example the non-constant equation of the form

$$\left(xy^2 ** x^2y^3\right) u_{xx} + 2\left(xy^2 ** x^3y^4\right) u_{xy} + 2\left(xy^2 ** x^4y^5\right) u_{yy} = \sin(x+y) ** \mathrm{e}^{x+y} \tag{2.31}$$

and we calculate the coefficients of Eq. (2.31) by using Eq. (2.27), (2.28) and (2.29), we obtain

$$D(x, y) = -\frac{31}{635\,040\,000} x^{10} t^{14}. \tag{2.32}$$

Then it is easy to see that Eq. (2.32) is always negative for all $(x_0, y_0) \in \mathbb{R}^2$, and thus Eq. (2.31) is an elliptic equation.

(ii): Suppose that $s + t, \alpha + \beta$ and $k + l$ are even and Eq. (2.26) is to be an elliptic equation under the condition that the powers $s = \frac{\alpha+k}{2}$ and $t = \frac{\beta+l}{2}$; also the coefficients of the two polynomials $a(x, y), c(x, y)$ have the same sign. Like in the previous section, for the classification of equation (2.26), we are using Eq. (2.30). The powers of x and y in Eq. (2.26) are even; thus for all points (x_0, y_0) in the domain $\mathbb{R}^2$, Eq. (2.26) is an elliptic equation. In particular, if we consider a non-constant equation of the form

$$\left(x^5y^7 ** xy^3\right) u_{xx} + 2\left(x^5y^7 ** xy^4\right) u_{xy} + 2\left(x^5y^7 ** xy^5\right) u_{yy} = \sin(x+y) ** \mathrm{e}^{x+y}, \tag{2.33}$$

in a similar way we obtain

$$D(x, y) = -\frac{1}{179\,805\,225\,600} x^{14} t^{24}. \tag{2.34}$$

Then it is easy to see that Eq. (2.34) is negative for all $(x_0, y_0) \in \mathbb{R}^2$; thus Eq. (2.33) is an elliptic equation.

3. Conclusion

We note that the classifications of hyperbolic and elliptic equations with non-constant coefficients are similar to those of the original equations after using double convolutions; that is, the equations are invariant under convolutions.

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